Seat No. ____________________

HK-003-2014008

B. Sc. (Sem. IV) (CBCS) (W.E.F. 2019) Examination

April - 2023

Mathematics : MATH - 04 (A)

(Linear Algebra, Real Analysis & Differential Geometry)

Faculty Code : 003

Subject Code : 2014008

Time : $2\frac{1}{2}$ Hours / Total Marks : **70**

Instructions : (1) All questions are compulsory.

(2) Figures to the right indicate full marks of the question.

1 (a) Answer the following questions : **4**

(1) True or False : Every bounded sequence is convergent.

(2) Find all the limit points of the sequence

$a_n = (-1)^n, n \in \mathbb{N}$.

(3) State Sandwich theorem for sequence.

(4) Define : Monotonically increasing sequence.

(b) Attempt any **one** : **2**

(1) Using Sandwich theorem show that the sequence $\{b_n\}$ converges to 0, where

$$b_n = \left[\frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + \ldots\ldots + \frac{1}{(2n)^2}\right]$$

(2) State Cauchy's General Principle of convergence of sequence.

(c) Attempt any **one** : **3**

(1) Let $\lim_{n\to\infty} a_n = l$. If $a_n > 0$ for all n, then show that

$$\lim_{n\to\infty} \left(a_1 \cdot a_2 \cdot .... \cdot a_n\right)^{1/n} = l.$$

(2) Show that : If $\lim_{n\to\infty} a_n = l$ and $a_n \geq 0$ for all n, then $l \geq 0$.

(d) Attempt any **one** : **5**

(1) Let $\{a_n\}$ be a sequence. Show that if $\lim_{n\to\infty} a_n = l$, then $\lim_{n\to\infty} \left(\frac{a_1 + a_2 + ... + a_n}{n}\right) = l$.

(2) Show that the sequence $\{S_n\}$ defined by the recursion formula $S_1 = 1$ and $S_{n+1} = \sqrt{3S_n}$, $n \geq 1$, converges to 3.

2 (a) Answer the following questions : **4**

(1) True or False : If $\lim_{n\to\infty} a_n = 0$, then $\sum a_n$ is convergent.

(2) State D'Alembert's Ratio test.

(3) Define : Sequence of Partial Sum of a series.

(4) Define : Geometric Series.

(b) Attempt any **one** : **2**

(1) Let $|r| < 1$. Show that

$$a - ar + ar^2 - ar^3 + ar^4 - ... = \frac{a}{1+r}.$$

(2) Discuss the convergence of the series :

$$\sqrt{\frac{1}{4}} + \sqrt{\frac{2}{6}} + \sqrt{\frac{3}{8}} + ... + \sqrt{\frac{n}{2n+2}} + ...$$

(c) Attempt any **one** : **3**

(1) Examine the convergence of the series :

$$\sum_{n=1}^{\infty}\left(\sqrt{n^2+1}-n\right).$$

(2) Examine the convergence of the series :

$$5-\frac{10}{3}+\frac{20}{9}-\frac{40}{27}+\ldots.$$

(d) Attempt any **one** : **5**

(1) Examine the convergence of the series :

$$\frac{3}{7}+\frac{3\cdot 6}{7\cdot 10}+\frac{3\cdot 6\cdot 9}{7\cdot 10\cdot 13}+\ldots..$$

(2) Examine the convergence of the series :

$$\sum_{n=1}^{\infty}\frac{n\cdot 2^n\cdot (n+1)!}{3^n\cdot n!}.$$

3 (a) Answer the following questions : **4**

(1) True or False : $T:\mathbb{R}^3\to\mathbb{R}^3$ defined by $T(x, y, z)=\left(x^2, xy^2, yz\right)$ is not a linear transformation.

(2) Truev or False : Identity linear transformation is idempotent.

(3) Define : Nilpotent linear transformation.

(4) Define : Zero linear transformation.

(b) Attempt any **one** : **2**

(1) Show that $T:\mathbb{R}\to\mathbb{R}^3$ defined as $T(x)=(x, 2x, 3x)$ is a linear transformation.

(2) Prove that $T:\mathbb{R}^3\to\mathbb{R}^3$ defined as

$T(x)=(x+y+z, y-z, x+2z)$ is singular linear transformation.

(c) Attempt any **one** : **3**

(1) Find rank and nullity of linear transformation $T:\mathbb{R}^2\to\mathbb{R}^3$ defined as $T(x_1, x_2)=(x_1, x_1+x_2, x_2)$.

(2) Let $T:U\to V$ be a non singular map. Prove that $T^{-1}:V\to U$ is a linear, one-one and onto.

(d) Attempt any **one** : **5**

(1) State and prove Rank-Nullity Theorem.

(2) Let *V* and *W* be two vector spaces. Let $\{v_1, v_2, ... v_n\}$ be a basis of *V*. Let $\{w_i, 1\le i\le n\}$ be any set of (not necessarily distinct) vectors in *W*. Prove that there is a unique linear map $T:V\to W$ such that $T(v_i)=w_i$.

4 (a) Answer the following questions : **4**

(1) Define : Linear functional.

(2) Define : Adjoint of a linear map.

(3) True or False : The set V^* of all linear functionals on a vector space $V(F)$ is also a vector space over the field *F*.

(4) True or False : A linear map associated with identity matrix with respect to standard bases is constant.

(b) Attempt any **one** : **2**

(1) Let a linear map $T : P_3(R) \to P_2(R)$ be defined by $T(p(x)) = p'(x)$. Find $[T : B_1, B_2]$, where $B_1 = \{1, x, x^2, x^3\}$ and $B_2 = \{1, x, x^2\}$.

(2) Define eigen value and eigen vector of linear map.

(c) Attempt any **one** : **3**

(1) Find a linear map associated with matrix

$$A = \begin{bmatrix} 2 & -3 & 4 \\ 1 & 0 & -1 \\ -2 & 1 & 0 \\ 1 & 2 & -2 \end{bmatrix}$$ with respect to standard bases

$B_1 = \{e_1, e_2, e_3\}$ and $B_2 = \{f_1, f_2, f_3, f_4\}$ for $\mathbb{R}^3$ and $\mathbb{R}^4$ respectively.

(2) Let a linear map $T : \mathbb{R}^3 \to \mathbb{R}^2$ be defined by

$T(e_1) = 2 \cdot f_1 + 0 \cdot f_2 + f_3, T(e_2) = 6 \cdot f_1 - \frac{3}{2} f_2 + \frac{11}{2} f_3,$

$T(e_3) = 0 \cdot f_1 - \frac{1}{4} f_2 - \frac{5}{4} f_3$, then find $[T : B_1, B_2]$.

(d) Attempt any **one** : **5**

(1) Find eigen values and eigen vectors for the linear map $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined as

$T(x, y, z) = (-2y - 2z, -2x - 3y - 2z, 3x + 6y + 5z)$

by considering the standard basis of $\mathbb{R}^3$.

(2) Let V be n dimensional vector space over the field R. If α is non-zero vector in V, then prove that there exists $f \in V^*$ such that $f(\alpha) \neq 0$.

5 (a) Answer the following questions : **4**

(1) True or False : Curvature is the reciprocal of radius of curvature.

(2) Find the curvature of the parabola $y = x^2$.

(3) Write formula of radius of curvature in polar form.

(4) Define : Cusp.

(b) Attempt any **one** : **2**

(1) Find the asymptotes parallel to coordinate axes of the curve $4x^2 + 9y^2 = x^2y^2$.

(2) Prove that (2, 0) is a node of the curve $y^2 = (x-1)(x-2)^2$.

(c) Attempt any **one** : **3**

(1) Find the points of inflection and the intervals in which the curve $y = x^3 - 3x^2 + 6x + 5$ is concave upward or concave downward.

(2) Find the radius of curvature at the origin of the curve $x^2y + xy^2 + xy + y^2 - 3x = 0$, using Newton's method.

(d) Attempt any **one** : **5**

(1) Find all the asymptotes of the curve

$$x^3 - x^2y - xy^2 + y^3 + 2x^2 - 4y^2 + 2xy + x + y + 1 = 0 \cdot$$

(2) (a) Find the radius of curvature of the curve

$x = a\left(\cos t - t\sin t\right),\ y = a\left(\sin t - t\cos t\right)$ at

$t = \dfrac{\pi}{4}$.

(b) Prove that $y = \log x$ is convex upward.